CURVES AND SURFACES: FINAL EXAM

This exam is of **50 marks** and is **3 hours long** - from 10 am to 1pm. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly.

1. Recall that a **generalised helix** is a space curve such that the curvature $\kappa \neq 0$ and all of its tangent vectors make a constant angle with a fixed direction $\mathbf{A}$. Let α be a generalised helix and β the curve optained by projecting α on to a plane orthogonal to $\mathbf{A}$. Let κ_*, τ_* be the curvature and torsion of the curves $*$.

- Prove that $\tau_\alpha/\kappa_\alpha$ is constant. (3)
- Prove that the principle normals to α and β are parallel. (3)
- Calculate κ_β in terms of κ_α. (4)

2. Let M be a surface in $\mathbb{R}^3$. Prove or give a counterexample to the following statements: Recall that a curve α on M is an **asymptotic curve** if $II_P((T(P), T(P)) = 0$ where $T(P)$ is the tangent direction at P, for all points P on α. Here II_P is the second fundamental form of M.

- A curve is both planar and an asymptotic curve if and only if it is a line. (3)
- A curve which is both an asymptotic curve and a line of curvature must be planar. (3)
- A curve is both a geodesic and an asymptotic curve if and only if it is a line. (3)
- A curve which is both a geodesic and a line of curvature must be planar. (3)

3. Let $\mathbf{M}_r$ be the surface given by the parametrization

$$x(u,v) = (\operatorname{sech}(u)\cos(v), \operatorname{sech}(u)\sin(v), u - \tanh(u))$$

where $-r \leq u \leq r$, $0 \leq v \leq 2\pi$. Let $\partial(\mathbf{M}_r)$ denote the boundary circles at $u = \pm r$.
Compute:

- The First Fundamental form I_P. (2)
- The Second Fundamental form II_P. (2)
- The matrix of the Shape Operator S_P. (2)
- The Mean curvature H. (2)
- The Gaussian curvature K. (2)
- The surface area of $\mathbf{M}_r$. What happens as $r \to \infty$ (3)
- The integral $\int_{\partial(\mathbf{M}_r)} \kappa_g ds$ where κ_g is the geodesic curvature of the boundary circles. What happens as $r \to \infty$? (3)
- The Total curvature $\iint_{\mathbf{M}_r} K dA$. What happens as $r \to \infty$? (3)
- The Euler characteristic $\chi(\mathbf{M}_r)$. What happens as $r \to \infty$? (3)

4a. Does there exist a **compact minimal surface** in $\mathbb{R}^3$? If yes, give an example. If no, prove it. (3)

4b. Does there exist a **compact flat surface**? If yes, give an example, if no, prove it. (3)

Some possibly useful formulae

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \text{ and } II = \begin{pmatrix} \ell & m \\ m & n \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{pmatrix} = I^{-1} \begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix}, \qquad \begin{pmatrix} \Gamma^u_{uv} \\ \Gamma^v_{uv} \end{pmatrix} = I^{-1} \begin{pmatrix} \frac{1}{2}E_v \\ \frac{1}{2}G_u \end{pmatrix}, \qquad \begin{pmatrix} \Gamma^u_{vv} \\ \Gamma^v_{vv} \end{pmatrix} = I^{-1} \begin{pmatrix} F_v - \frac{1}{2}G_u \\ \frac{1}{2}G_v \end{pmatrix}$$

$$\ell_v - m_u = \ell \Gamma uv^u + m(\Gamma^v_{uv} - \Gamma^u_{uu}) - n\Gamma^v_{uu}$$

$$m_v - n_u = \ell \Gamma vv^u + m(\Gamma^v_{vv} - \Gamma^u_{uv}) - n\Gamma^v_{uv}$$